HYPERELLIPTICITY AND SYSTOLES OF KLEIN SURFACES

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Abstract. Given a hyperelliptic Klein surface, we construct companion Klein bottles, extending our technique of companion tori already exploited by the authors in the genus 2 case. Bavard's short loops on such companion surfaces are studied in relation to the original surface so to improve a systolic inequality of Gromov's. A basic idea is to use length bounds for loops on a companion Klein bottle, and then analyze how curves transplant to the original non-orientable surface. We exploit the real structure on the orientable double cover by applying the coarea inequality to the distance function from the real locus. Of particular interest is the case of Dyck's surface. We also exploit an optimal systolic bound for the Möbius band, due to Blatter.

Contents

1. Introduction

Systolic inequalities for surfaces compare length and area, and can therefore be thought of as "opposite" isoperimetric inequalities. The

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study of such inequalities was initiated by C. Loewner in 1949 when he proved his torus inequality for $\mathbb{T}^2$ (see Pu [42] and Horowitz *et al.* [27]). The systole, denoted "sys", of a space is the least length of a loop which cannot be contracted to a point in the space, and is therefore a natural generalisation of the *girth* invariant of graphs.

Theorem 1.1 (Loewner's torus inequality). *Every metric on the* 2-*dimensional torus* $\mathbb{T}$ *satisfies the bound*

$$\operatorname{sys}(\mathbb{T}) \leq 2^{\frac{1}{2}} 3^{-\frac{1}{4}} \sqrt{\operatorname{area}(\mathbb{T})}. \tag{1.1}$$

In higher dimensions, M. Gromov's deep result [21], relying on filling invariants, exhibits a universal upper bound for the systole in terms of the total volume of an essential manifold. L. Guth [26] recently found an alternative proof not relying on filling invariants, and giving a generalisation of Gromov's inequality, see also Ambrosio and Katz [1].

In dimension 2, the focus has been, on the one hand, on obtaining near-optimal asymptotic results in terms of the genus [32, 34], and on the other, on obtaining realistic estimates in cases of low genus [33, 27]. One goal has been to determine whether all aspherical surfaces satisfy Loewner's bound (1.1), a question that is still open in general. We resolved it in the affirmative for genus 2 in [33]. An optimal inequality of C. Bavard [6] for the Klein bottle K is stronger than Loewner's bound:

$$\operatorname{sys}(K) \leq C_{\text{Bavard}} \sqrt{\operatorname{area}(K)}, \qquad C_{\text{Bavard}} = \pi^{\frac{1}{2}} 8^{-\frac{1}{4}} \approx 1.0539. \tag{1.2}$$

Gromov proved a general estimate for all aspherical surfaces:

$$\operatorname{sys}^2 \leq \frac{4}{3} \operatorname{area}, \tag{1.3}$$

see [21, Corollary 5.2.B]. As Gromov points out in [21], the $\frac{4}{3}$ bound (1.3) is actually *optimal* in the class of Finsler metrics. Therefore any further improvement is not likely to result from a mere application of the coarea formula. One can legitimately ask whether any improvement is in fact possible, of course in the framework of Riemannian metrics.

The goal of the present article is to furnish such an improvement in the case of non-orientable surfaces. We will say that such a surface is hyperelliptic if its orientable double cover is.

Theorem 1.2. *Let* $n \geq 2$. *Every Riemannian metric from a hyperelliptic conformal type on the surface* $n\mathbb{RP}^2$ *satisfies the bound*

$$\operatorname{sys}^2 \leq 1.333 \ \operatorname{area}. \tag{1.4}$$

Note the absence of an ellipsis following "333", making our estimate an improvement on Gromov's $\frac{4}{3}$ bound (1.3). By keeping track of the best constants in our estimates throughout the proof of the theorem, one could obtain a slightly better bound. However, our goal is merely to develop techniques sufficient to improve the $\frac{4}{3}$ bound. Since every conformal class on Dyck's surface $K\#\mathbb{RP}^2 = \mathbb{T}^2\#\mathbb{RP}^2 = 3\mathbb{RP}^2$ of Euler characteristic -1 is hyperelliptic, we have

Corollary 1.3. *Every Riemannian metric on Dyck's surface* $3\mathbb{RP}^2$ *satisfies*

$$\mathrm{sys}(3\mathbb{RP}^2)^2 \leq 1.333 \ \mathrm{area}(3\mathbb{RP}^2).$$

The proof of the main theorem exploits a variety of techniques ranging from hyperellipticity to the coarea formula and cutting and pasting. The current best upper bound for the systole on Dyck's surface only differs by about 30% from the best known example given by a suitable extremal hyperbolic surface (see Section 4).

We will exploit the following characterisation of the systole of a non-orientable surface. Given a metric on a Klein surface $X = \tilde{X}/\tau$ where $\tau : \tilde{X} \to \tilde{X}$ is fixed point-free, we consider the natural τ-invariant pullback metric on its orientable double cover $\tilde{X}$.

Definition 1.4. The least displacement "disp" of $\tau : \Sigma_g \to \Sigma_g$ is the number

$$\mathrm{disp}(\tau) = \min \left\{ \mathrm{dist}(x, \tau(x)) \mid x \in \Sigma_g \right\}. \tag{1.5}$$

Proposition 1.5. *The systole of a Klein surface X can be expressed as the least of the following two quantities:*

$$\mathrm{sys}(X) = \min \left\{ \mathrm{sys}(\tilde{X}), \mathrm{disp}(\tau) \right\}.$$

Indeed, lifting a systolic loop of X to $\tilde{X}$, we obtain either a loop in the orientable cover Σ_g, or a path connecting two points which form an orbit of the deck transformation τ.

Recall that the following four properties of a closed surface Σ are equivalent: (1) Σ is aspherical; (2) the fundamental group of Σ is infinite; (3) the Euler characteristic of Σ is non-positive; (4) Σ is not homeomorphic to either S^2 or $\mathbb{RP}^2$. The following conjecture has been discussed in the systolic literature, see [30].

Conjecture 1.6. *Every aspherical surface satisfies Loewner's bound*

$$\frac{\mathrm{sys}^2}{\mathrm{area}} \leq \frac{2}{\sqrt{3}}. \tag{1.6}$$

M. Gromov [21] proved an asymptotic estimate which implies that every orientable surface of genus greater than 50 satisfies Loewner's bound. This was extended to orientable surfaces of genus at least 20 in [32], and for the genus 2 surface in [33].

Recent publications in systolic geometry include Ambrosio & Katz [1], Babenko & Balacheff [2], Balacheff *et al.* [3], Belolipetsky [5], El Mir [16], Fetaya [18], Katz *et al.* [28, 29, 35], Makover & McGowan [38], Parlier [41], Ryu [43], and Sabourau [44].

In Section 2, we will review the relevant conformal information, including hyperellipticity. In Section 3, we will present metric information in the context of a configuration of the five surfaces appearing in our main argument. Section 4 exploits optimal inequalities of Blatter and Sakai for the Möbius band so to prove our first theorem for $n = 3$. We handle the remaining case, namely $n \geq 4$, in Section 5.

2. Review of conformal information and hyperellipticity

In this section we review the necessary pre-metric (i.e., conformal) information. The quadratic equation $y^2 = p$ over $\mathbb{C}$ is well known to possess two distinct solutions for every $p \neq 0$, and a unique solution for $p = 0$. Now consider the locus (solution set) of the equation

$$y^2 = p(x) \tag{2.1}$$

for $(x, y) \in \mathbb{C}^2$ and generic $p(x)$ of even degree $2g + 2$. Such a locus defines a Riemann surface which is a branched two-sheeted cover of $\mathbb{C}$. The cover is obtained by projection to the x-coordinate. The branching locus corresponds to the roots of $p(x)$.

There exists a unique smooth closed Riemann surface Σ_g naturally associated with (2.1), sometimes called the *smooth completion* of the affine surface (2.1), together with a holomorphic map

$$Q_g : \Sigma_g \to \hat{\mathbb{C}} = S^2 \tag{2.2}$$

extending the projection to the x-coordinate. By the Riemann-Hurwitz formula, the genus of the smooth completion is g, where $\deg(p(x)) = 2g - 2$. All such surfaces are hyperelliptic by construction. The hyperelliptic involution $J : \Sigma_g \to \Sigma_g$ flips the two sheets of the double cover of S^2 and has exactly $2g + 2$ fixed points, called the Weierstrass points of Σ_g. The hyperelliptic involution is unique. The involution J can be identified with the nontrivial element in the center of the (finite) automorphism group of X (*cf.* [17, p. 108]) when it exists, and then such a J is unique [39, p.204].

A hyperelliptic closed Riemann surface Σ_g admitting an orientation-reversing (antiholomorphic) involution τ can always be reduced to the

form (2.1) where $p(x)$ is a polynomial all of whose coefficients are real, where the involution $\tau : \Sigma_g \to \Sigma_g$ restricts to complex conjugation on the affine part of the surface in $\mathbb{C}^2$, namely

$$\tau(x, y) = (\bar{x}, \bar{y}). \tag{2.3}$$

The special case of a fixed point-free involution τ can be represented as the locus of the equation

$$-y^2 = \prod_j (x - x_j)(x - \bar{x}_j), \tag{2.4}$$

where $x_j \in \mathbb{C} \setminus \mathbb{R}$ for all j. Here the minus sign on the left hand side ensures the absence of real solutions, and therefore the fixed point-freedom of τ. The uniqueness of the hyperelliptic involution implies the following.

Proposition 2.1. *We have the commutation relation $\tau \circ J = J \circ \tau$.*

A Klein surface X is a non-orientable closed surface. Such a surface can be thought of as an antipodal quotient $X = \Sigma_g/\tau$ of an orientable surface by a fixed point-free, orientation-reversing involution τ. The pair (Σ_g, τ) is known as a real Riemann surface. A Klein surface is homeomorphic to the connected sum

$$X = n\mathbb{RP}^2 = \underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{n},$$

of n copies of the real projective plane. The case $n = 2$ corresponds to the Klein bottle $K = 2\mathbb{RP}^2$, covered by the torus. In the case $n = 3$, we obtain the surface $3\mathbb{RP}^2$ of Euler characteristic -1, whose orientable double cover is the genus 2 surface Σ_2.

In the sequel, we will focus on Dyck's surface $3\mathbb{RP}^2$ since the proof of the main theorem in this case requires special arguments.

We will denote by $\tilde{X}$ the orientable double cover of a Klein surface X. If a surface X is hyperelliptic, we will denote by $\mathrm{Br}(X) \subset \hat{\mathbb{C}}$ its branch locus, i.e., the set of isolated branch points of the double cover $X \to \hat{\mathbb{C}}$.

Definition 2.2. A Klein bottle K is called a *companion* of a Klein surface X if we have the inclusion of the branch loci at the level of the orientable double covers:

$$\mathrm{Br}(\tilde{K}) \subset \mathrm{Br}(\tilde{X}) \subset \hat{\mathbb{C}}.$$

Proposition 2.3. *Each Klein surface K homeomorphic to $3\mathbb{RP}^3$ admits a triplet of companion Klein bottles K_1, K_2, K_3 satisfying the following three conditions:*

1. $|\mathrm{Br}(\tilde{K}_j)| = 4$*;*

(2) $|\mathrm{Br}(\tilde{K}_i) \cap \mathrm{Br}(\tilde{K}_j)| = 2$ *for* $i \neq j$*;*
(3) $\cup_{j=1}^{3} \mathrm{Br}(\tilde{K}_j) = \mathrm{Br}(\tilde{X})$.

Proof. Given a real Riemann surface (Σ_g, τ), consider the presentation (2.4) with $p(x)$ a real polynomial. We can write the roots of p as a collection of conjugate pairs $(a, \bar{a})$. Thus in the genus 2 case, the affine form of the surface is the locus of the equation

$$-y^2 = (x-a)(x-\bar{a})(x-b)(x-\bar{b})(x-c)(x-\bar{c}) \tag{2.5}$$

in $\mathbb{C}^2$. Choosing two conjugate pairs, for instance $(a, \bar{a}, b, \bar{b})$, we can construct a companion surface

$$-y^2 = (x-a)(x-\bar{a})(x-b)(x-\bar{b}), \tag{2.6}$$

By the Riemann-Hurwitz formula, its genus is one, and therefore the (smooth completion of the) companion surface is a torus. We will denote it $\mathbb{T}_{a,b}$. By construction, its set of zeros is τ-invariant. In other words, the (affine part in $\mathbb{C}^2$ of the) torus is invariant under the action of complex conjugation. Thus, the surface $\mathbb{T}_{a,b}/\tau$ is a Klein bottle K, namely, a companion Klein bottle of the original Klein surface $3\mathbb{RP}^2 = \Sigma_2/\tau$ stemming from (2.5). We thus obtain the three Klein bottles $\mathbb{T}_{a,b}/\tau$, $\mathbb{T}_{b,c}/\tau$, and $\mathbb{T}_{c,a}/\tau$, proving the proposition. □

The maps constructed so far can be represented by the following diagram of homomorphisms (note that two out of the four arrows point in the leftward direction):

$$3\mathbb{RP}^2 \leftarrow \Sigma_2 \rightarrow S^2 \leftarrow \mathbb{T}_{a,b} \rightarrow K. \tag{2.7}$$

Complex conjugation τ on $\hat{\mathbb{C}} = S^2$ fixes a circle called the equatorial circle, denoted $\hat{\mathbb{R}} \subset \hat{\mathbb{C}}$.

Definition 2.4. The *equator* of Σ_2 is the circle given by the inverse image of the equator $\hat{\mathbb{R}}$ under the hyperelliptic projection $\Sigma_2 \to S^2$.

Lemma 2.5. *The equator of* Σ_2 *coincides with the fixed point set of the composition* $\tau \circ J$*. The equator is invariant under the action of* τ*. The action of* τ *on the equator of* Σ_2 *is fixed point-free.*

Proof. The lemma is immediate from Proposition 2.1. □

Similarly, we obtain the following.

Lemma 2.6. *Relative to the double cover* $\mathbb{T}_{a,b} \to \hat{\mathbb{C}}$*, the inverse image of the equatorial circle* $\hat{\mathbb{R}} \subset \hat{\mathbb{C}}$ *is a pair of disjoint circles, the involution* τ *acts on the torus by switching the two circles, while the involution* $\tau \circ J$ *fixes both circles pointwise.*

A loop on a non-orientable surface is called 2-sided if its tubular neighborhood is homeomorphic to an annulus, and 1-sided if its tubular neighborhood is homeomorphic to a Möbius band. The following lemma is immediate from the homotopy lifting property.

Lemma 2.7. *Let X be a Klein surface, and $\tilde{X}$ its orientable double cover. Let $\gamma \subset X$ be a one-sided loop, and $\delta \subset X$ a 2-sided loop in $X = \tilde{X}/\tau$. We have the following four properties:*

1. *Lifting γ to $\tilde{X}$ yields a path on $\tilde{X}$ connecting a pair of points which form an orbit of the involution τ;*
2. *lifting δ to $\tilde{X}$ yields a closed curve on X;*
3. *the inverse image of γ under the double cover $\tilde{X} \to X$ is a circle (i.e. has a single connected component homeomorphic to a circle);*
4. *the inverse image of δ under the double cover $\tilde{X} \to X$ has a pair of connected components (circles).*

The following blend of topological and hyperelliptic information will be helpful in the sequel. The upperhalf plane in $\mathbb{C}$ is a fundamental domain for the action of complex conjugation τ. Hence points on the Klein surface $3\mathbb{RP}^2 = \Sigma_2/\tau$ can be represented by points in the closure of the upperhalf plane. Consider the northern hemisphere

$$\hat{\mathbb{C}}^+ \subset \hat{\mathbb{C}} = S^2,$$

with the equator included. We will think of the surface $3\mathbb{RP}^2$ as a double cover

$$3\mathbb{RP}^2 \to \hat{\mathbb{C}}^+. \tag{2.8}$$

The double cover (2.8) is branched along the equator as well as at three additional Weierstrass points, corresponding to the points a, b, c of the usual hyperelliptic cover $\Sigma_2 \to \hat{\mathbb{C}}$. We also have an analogue of the hyperelliptic involution, namely the deck transformation

$$J : 3\mathbb{RP}^2 \to 3\mathbb{RP}^2, \tag{2.9}$$

fixing the equator and the three Weierstrass points. Note that the three remaining Weierstrass points $\bar{a}, \bar{b}, \bar{c} \in \Sigma_2$ are mapped to a, b, c by the involution τ.

Lemma 2.8. *A simple loop $\Delta \subset \hat{\mathbb{C}}^+$ parallel to the equator decomposes the northern hemisphere into a union of a disk D "north" of Δ and an annulus A "south" of Δ:*

$$\hat{\mathbb{C}}^+ = D \cup_\Delta A, \tag{2.10}$$

where $a, b, c \in D$ (all the branch points of $\hat{\mathbb{C}}^+$ are in the disk D). The corresponding decomposition of $3\mathbb{RP}^2$ is

$$3\mathbb{RP}^2 = \Sigma_{1,1} \cup_{S^1} \mathrm{Mob}, \tag{2.11}$$

where $\Sigma_{1,1}$ is the once-holed torus, and Mob, *the Möbius band.*

We will refer to this decomposition as the annulus decomposition, since it is the non-orientable analogue of the decomposition (2.10), see Figure 2.1. In terms of the connected sum notation, the decomposition (2.11) corresponds to the topological decomposition $3\mathbb{RP}^2 = \mathbb{T}^2 \# \mathbb{RP}^2$.

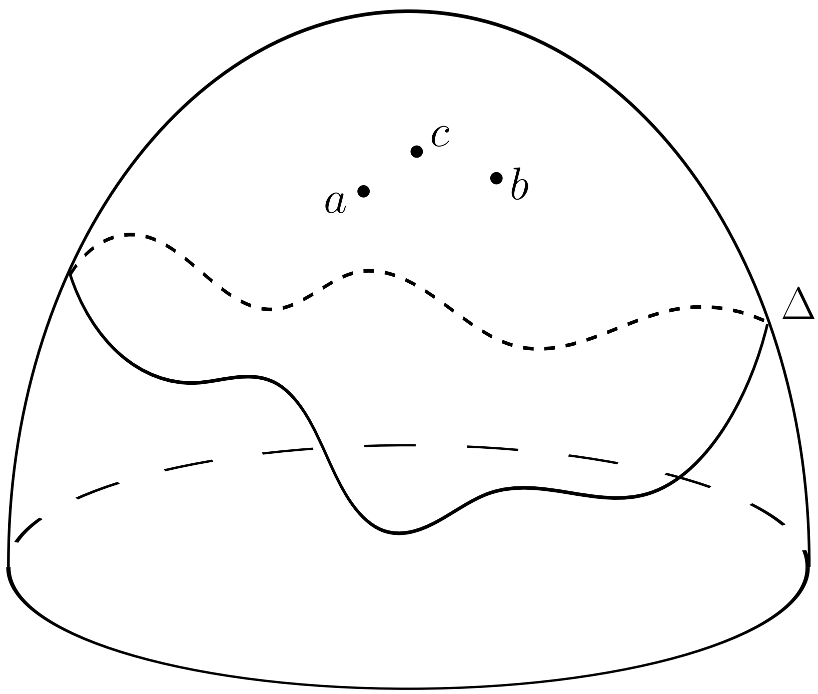

FIGURE 2.1. Klein surface as a hemispherical double cover

The main idea in the proof of the main theorem is to reduce the general situation to a case of a *metrically controlled* annulus decomposition of the Klein surface, as explained in Proposition 3.1 below.

3. Reduction to the annulus decomposition

A noncontractible loop on $3\mathbb{RP}^2$ satisfying Bavard's bound (1.2) will be called a Bavard loop.

Proposition 3.1. *The Klein surface $X = 3\mathbb{RP}^2$ either contains a Bavard loop, or admits a simple loop $\delta \subset X$ which separates it as follows:*

$$X = \Sigma_{1,1} \cup_\delta \mathrm{Mob},$$

such that moreover

(1) *the surface $\Sigma_{1,1}$ contains the three Weierstrass points;*
(2) *the Möbius band* Mob *contains the equatorial circle;*
(3) *the loop $\delta \subset X$ is J-invariant;*
(4) *the loop δ is of length at most $2C_{\mathrm{Bavard}}\sqrt{\mathrm{area}}$;*
(5) *the loop δ double-covers a loop $\Delta \subset \hat{\mathbb{C}}^+$ which lifts to a systolic loop on a companion torus.*

The proposition will be proved in this section. We start with some preliminary observations. Pulling back the metric to a double cover results in a metric of twice the area. We therefore obtain the following lemma.

Lemma 3.2. *Given a J-invariant metric on the Klein surface $3\mathbb{RP}^2$, we have the following relations among the areas of the surfaces appearing in diagram* (2.7)*:*

$$\mathrm{area}(3\mathbb{RP}^2) = \mathrm{area}(S^2) = \mathrm{area}(K),$$

as well as the relation $\mathrm{area}(\Sigma_2) = \mathrm{area}(\mathbb{T}_{a,b}) = 2\,\mathrm{area}(3\mathbb{RP}^2)$.

Next, we show that averaging the metric improves the systolic ratio.

Proposition 3.3. *Let Σ be a hyperelliptic orientable double cover of a Klein surface X. Consider a Riemannian metric on Σ. Then*

(1) *averaging its metric by the hyperelliptic involution J increases the systolic ratio $\frac{\mathrm{sys}^2}{\mathrm{area}}$ of Σ;*
(2) *averaging its metric by the complex conjugation τ increases the systolic ratio $\frac{\mathrm{sys}^2}{\mathrm{area}}$ of Σ;*
(3) *the ratio $\frac{\mathrm{disp}(\tau)^2}{\mathrm{area}}$ defined in* (1.5) *increases under averaging.*

Proof. This point was discussed in detail in [4]. We summarize the argument as follows. Express the metric in terms of a constant curvature metric in its conformal class, by means of a conformal factor f^2. Thus, in the case of a torus we obtain a metric $f^2(p)(dx^2 + dy^2)$ at a typical point p of the torus, where the function f is doubly periodic. We average the factor f^2 by the hyperelliptic involution $J : \Sigma \to \Sigma$, i.e., we replace $f^2(p)$ by

$$\frac{1}{2}\left(f^2(p) + f^2(J(p))\right).$$

Such averaging preserves the total area of the metric. Similarly, it preserves the energy of a curve on the surface. Choosing a constant speed parametrisation of a systolic loop for the original metric, we see that its energy is preserved under averaging. Hence its length is not decreased by averaging. Similar remarks apply in the two remaining cases. □

Given a Klein surface $X = 3\mathbb{RP}^2$, we can similarly average the metric by the hyperelliptic involution (2.9). Hence we may assume without loss of generality that the metrics on both $3\mathbb{RP}^2$ and Σ_2 are J-invariant.

We now consider a companion Klein bottle K as in Proposition 2.3. We will seek to transplant short loops from K to X. A systolic loop on K is either a 1-sided loop γ, or a 2-sided loop δ (see Lemma 2.7).

Now consider the real model (2.5) of Σ_2, and its three companion tori of type (2.6). For each companion torus, we pass to the quotient Klein bottle, and find a systolic loop satisfying Bavard's bound. We thus obtain three loops $\delta_{a,b}$, $\delta_{b,c}$, and $\delta_{a,c}$. If all three are two-sided, they lift to loops $\tilde{\delta}_{a,b} \subset \mathbb{T}_{a,b}$, $\tilde{\delta}_{b,c} \subset \mathbb{T}_{b,c}$, and $\tilde{\delta}_{c,a} \subset \mathbb{T}_{c,a}$ on the tori. Let $\Delta_{a,b} \subset \hat{\mathbb{C}}$ be the projection of the loop $\tilde{\delta}_{a,b}$ to the sphere, and similarly for $\Delta_{b,c}$ and $\Delta_{c,a}$. Each loop $\Delta \subset \hat{\mathbb{C}}$ defines a partition of the 6-point set $\mathrm{Br}(\Sigma_2) \subset \hat{\mathbb{C}}$.

Proposition 3.4. *Assuming the loops $\delta_{a,b}$, etc., are 2-sided, build the corresponding loops $\Delta_{a,b} \subset \hat{\mathbb{C}}$, etc. Consider the three partitions of $\mathrm{Br}(\Sigma_2) \subset \hat{\mathbb{C}}$ defined by the three loops $\Delta \subset S^2$. If the three partitions are not identical, then there is a Bavard loop on Σ_2.*

Proof. Consider a Bavard systolic loop $\delta \subset K$ of a Klein bottle K. Consider its lift $\tilde{\delta} \subset \mathbb{T}$ to the torus, and the projection $\Delta \subset \hat{\mathbb{C}}$. If two such loops are non-homotopic in

$$S^2 \setminus \{\mathrm{Br}(\Sigma_2)\},$$

we apply the cut and paste technique of [33] to rearrange segments of the two loops into a pair of loops that lift to closed paths on the genus 2 surface. One of the lifts is necessarily Bavard. □

Proposition 3.5. *If a systolic loop on a companion Klein bottle is 1-sided, then we can transplant it to the Klein surface $3\mathbb{RP}^2$, which therefore satisfies Bavard's inequality.*

Proof. Given a genus 2 surface (2.5), consider a companion Klein bottle $K_{a,b} = \mathbb{T}_{a,b}/\tau$. Consider a systolic loop on $K_{a,b}$. If a systolic loop γ

is one-sided, then γ lifts to a path connecting a pair of points in an orbit of τ on the torus. The proof is completed by combining Lemma 3.6 and Lemma 3.7 below. □

Lemma 3.6. *Let $\gamma \subset K$ be a 1-sided loop, and let $\tilde{\gamma} \subset \mathbb{T}$ be the circle which is the connected double cover of γ. Then there is a pair of real points $p, \tau(p) \subset \tilde{\gamma}$ which decompose $\tilde{\gamma}$ into a pair of paths:*

$$\tilde{\gamma} = \gamma_+ \cup \gamma_-,$$

such that each of the paths γ_+, γ_- projects to a closed curve Γ_+, Γ_- on $\hat{\mathbb{C}}$.

Proof. Note that $\tilde{\gamma} \subset \mathbb{T}$ is invariant under the fixed point-free action of τ on the torus. The loop $\tilde{\gamma}$ projects to a loop denoted

$$\Gamma \subset \hat{\mathbb{C}}$$

under the hyperelliptic quotient $Q : \mathbb{T} \to \hat{\mathbb{C}}$. Let $\hat{\mathbb{R}} \subset \hat{\mathbb{C}}$ be the fixed point set of τ acting on $\hat{\mathbb{C}}$. The connected loop $\Gamma \subset \hat{\mathbb{C}}$ is invariant under complex conjugation. Therefore it must meet the equator $\hat{\mathbb{R}}$ in a point $p_0 \in \hat{\mathbb{R}}$ (see Section 2). Let

$$Q^{-1}(p_0) = \{p, \tau(p)\} \subset \mathbb{T}.$$

Note that p_0 is a self-intersection point of Γ. It may be helpful to think of Γ as a figure-eight loop, with its center-point p_0 on the equator $\hat{\mathbb{R}}$.

We now view the original loop $\gamma \subset K$ as a path starting at the image of the point p in K. We lift the path γ to a path $\gamma_+ \subset \mathbb{T}$ joining p and $\tau(p)$. The path γ_+ projects to half the loop $\Gamma \subset \hat{\mathbb{C}}$, forming one of the hoops of the figure-eight. If we let $\gamma_- = \tau(\gamma_+)$, we can write $\tilde{\gamma} = \gamma_+ \cup \gamma_-$. Thus the loop $\Gamma \subset \hat{\mathbb{C}}$ is the union of two loops $\Gamma = Q(\gamma_+) \cup Q(\gamma_-)$. □

The following lemma refers to the five surfaces appearing in diagram (2.7).

Lemma 3.7. *Let $\gamma_+ \subset \mathbb{T}_{a,b}$ be the path constructed in Lemma 3.6. Consider the loop $Q(\gamma_+) \subset \hat{\mathbb{C}}$, and lift it to a path $\tilde{\tilde{\gamma}} \subset \Sigma_2$. Then the image of $\tilde{\tilde{\gamma}}$ under the covering projection $\Sigma_2 \to 3\mathbb{RP}^2$ is a closed curve.*

Proof. To transplant γ_+ to the Klein surface $3\mathbb{RP}^2$, note that the path $\tilde{\tilde{\gamma}} \subset \Sigma_2$ may or may not close up, depending on the position of the third pair $(c, \bar{c})$ of branch points of $\Sigma_2 \to \hat{\mathbb{C}}$. If $\tilde{\tilde{\gamma}}$ is already a loop, then it projects to a Bavard loop on the Klein surface $3\mathbb{RP}^2 = \Sigma/\tau$. In the remaining case, the path $\tilde{\tilde{\gamma}}$ connects a pair of opposite points $\tilde{p}, \tau(\tilde{p})$ on the surface Σ_2, where $Q(\tilde{p}) = Q(\tau(\tilde{p})) = p_0$. Therefore $\tilde{\tilde{\gamma}}$ projects to a Bavard loop in this case, as well. □

By Proposition 3.5, it remains to consider the case when each systolic loop δ of each of the three companion Klein bottles $K_{a,b}, K_{a,c}, K_{b,c}$ is 2-sided. Thus each of these Bavard loops $\delta_{a,b}, \delta_{a,c}, \delta_{b,c}$ lifts to a closed curve $\tilde{\delta}$ on the corresponding torus. Let $\Delta = Q(\tilde{\delta}) \subset \hat{\mathbb{C}} = S^2$ be the corresponding loop on the sphere.

Lemma 3.8. *If Δ meets the equator, then the original Klein surface $3\mathbb{RP}^2$ contains a Bavard loop.*

Proof. Let $p_0 \in \Delta \cap \hat{\mathbb{R}}$. Let $p, \tau(p) \in \Sigma_2$ be the points above it in Σ_2. We lift the path Δ starting at p_0 to a path $\delta_+ \subset \Sigma_2$ starting at p. If δ_+ closes up, its projection to $3\mathbb{RP}^2$ is the desired Bavard loop. Otherwise, the path δ_+ connects p to $\tau(p)$. In this case as well, the path δ_+ projects to a Bavard loop on $3\mathbb{RP}^2 = \Sigma_2/\tau$. $\square$

It remains to consider the case when $\Delta \cap \hat{\mathbb{R}} = \emptyset$. This corresponds to the annulus decomposition case of Proposition 3.1, once we show that the loop is simple in the following lemma.

Lemma 3.9. *Let $\tilde{\delta}$ be a systolic loop on the torus, and let $\Delta = Q(\tilde{\delta}) \subset \hat{\mathbb{C}} = S^2$ be the corresponding loop on the sphere. Assume that $\Delta \cap \hat{\mathbb{R}} = \emptyset$. Then the loop Δ is simple.*

Proof. By hypothesis, the loop Δ lies in a hemisphere, i.e., one of the connected components of $\hat{\mathbb{C}} \setminus \hat{\mathbb{R}}$. The typical case of a non-simple loop to keep in mind is a figure-eight curve. Denote by $a, b \in \hat{\mathbb{C}}$ the branch points with respect to which Δ has odd winding number. We will think of the curve Δ as defining a connected graph $\mathcal{A} \subset \mathbb{R}^2$ in a plane. The vertices of the graph are the self-intersection points of Δ. Each vertex necessarily has valence 4. By adding the bounded "faces" to the graph, we obtain a "fat" graph $\mathcal{A}_{\mathrm{fat}}$ (the typical example is the interior of the figure-eight). More precisely, the complement $\mathbb{C} \setminus \mathcal{A}$ has a unique *unbounded* connected component, denoted $E \subset \mathbb{C} \setminus \mathcal{A}$. Its complement in the plane, denoted

$$\mathcal{A}_{\mathrm{fat}} = \mathbb{C} \setminus E,$$

contains both the graph $\mathcal{A}$ and its bounded "faces". Since Δ has odd winding number with respect to the branch points a, b of the double cover $Q : \mathbb{T}^2_{a,b} \to \hat{\mathbb{C}}$, they must both lie inside the connected region $\mathcal{A}_{\mathrm{fat}}$:

$$a, b \in \mathcal{A}_{\mathrm{fat}}.$$

The boundary of $\mathcal{A}_{\mathrm{fat}}$ can be parametrized by a closed curve ℓ, thought of the boundary of the outside component $E \subset \mathbb{C}$ so as to define an orientation on ℓ (in the case of the figure-eight loop, this results in reversing the orientation on one of the hoops of the figure-eight). Note

that $\ell \subset \mathcal{A}$ is a subgraph. Since both branch points lie inside, the loop ℓ had odd winding number with respect to each of the points a and b. Hence ℓ lifts to a noncontractible loop on the torus. If Δ is not simple, then the boundary of the outside region E is not smooth, i.e., the loop ℓ must contain "corners" and can therefore be shortened, contradicting the hypothesis that δ is a systolic loop. □

4. Improving Gromov's 3/4 bound

In this section, we will prove the main theorem for $n = 3$.

Theorem 4.1. *Let* $\beta = \sqrt{1.333} \simeq 1.1545$. *The bound*

$$\mathrm{sys} \leq \beta\sqrt{\mathrm{area}}$$

is satisfied by every metric on Dyck's surface $3\mathbb{RP}^2$.

The orientable double cover of the hyperbolic Dyck's surface $3\mathbb{RP}^2$ with the maximal systole was described by Parlier [40]. Silhol [46, 47] identified a presentation of its affine form:

$$y^2 = x^6 + ax^3 + 1, \quad a = 434 + 108\sqrt{17}.$$

See also Lelièvre & Silhol [37] and Gendulphe [20]. The maximal systole on a hyperbolic Dyck's surface $3\mathbb{RP}^2$ is equal to $\operatorname{arccosh} \frac{5+\sqrt{17}}{2} = 2.19\ldots$ resulting in a systolic ratio of $0.76\ldots$.

To prove Theorem 4.1, we use the partition $3\mathbb{RP}^2 = \Sigma_{1,1} \cup \mathrm{Mob}$, as constructed in Proposition 3.1. Since we are studying a scale-invariant systolic ratio, we may and will normalize $3\mathbb{RP}^2$ to unit area:

$$\mathrm{area}(3\mathbb{RP}^2) = 1.$$

By Proposition 3.3, we can assume that the metric on $3\mathbb{RP}^2$ is J-invariant.

By Proposition 3.1, our desired bound on the systolic ratio of the Klein surface $3\mathbb{RP}^2 = \Sigma_2/\tau$ reduces to the case when the Bavard loops δ on the companion Klein bottles are 2-sided, but their lifts $\tilde{\delta}$ to the triplet of tori $\mathbb{T}$ project to loops $\Delta \subset \hat{\mathbb{C}}$, where each of the three loops Δ defines the same partition of the set $\mathrm{Br}(\Sigma_2) \subset \hat{\mathbb{C}}$. By Lemma 3.8, we may assume each Δ lies in the open northern hemisphere $\hat{\mathbb{C}}^+$, and that its lift to Σ_2 produces a non-closed curve. Thus, we may assume that each of the simple loops $\Delta \subset \hat{\mathbb{C}}$ separates the six points $a, \bar{a}, b, \bar{b}, c, \bar{c}$ into two triplets (a, b, c) and $(\bar{a}, \bar{b}, \bar{c})$. Hence its connected double cover in Σ_2 is isotopic to the equatorial circle $Q^{-1}(\hat{\mathbb{R}}) \subset \Sigma_2$. The latter is a double cover of the equator $\hat{\mathbb{R}} \subset \hat{\mathbb{C}}$ (see Section 2).

We will start with a few preliminary topological results.

Lemma 4.2. *We have* $\mathrm{sys}(3\mathbb{RP}^2) \leq 2\,\mathrm{dist}(\mathrm{Br}(3\mathbb{RP}^2), Eq)$*, where* Eq *is the equator of* $\mathbb{C}^+$*.*

In particular, if $\mathrm{sys}(3\mathbb{RP}^2) \geq \beta$ *then*

$$\mathrm{dist}(\mathrm{Br}(3\mathbb{RP}^2), Eq) \geq \tfrac{\beta}{2}. \tag{4.1}$$

Proof. Every path on $\hat{\mathbb{C}}$ connecting a branch point to the equator is double covered by a noncontractible loop in $3\mathbb{RP}^2$. The lemma follows. □

Definition 4.3. Define $Y \subset \Sigma_2$ as the preimage of $\hat{\mathbb{C}}^+$ under the ramified cover $\Sigma_2 \to \hat{\mathbb{C}}$. Observe that Y is a punctured torus and that $3\mathbb{RP}^2 = Y/\tau$ by identifying the opposite points on ∂Y by τ.

Lemma 4.4. *If sys*$(3\mathbb{RP}^2) \geq \beta$ *then every arc of* Y *with endpoints in* ∂Y *of length less than* 0.6276 *is homotopically trivial in* $\pi_1(Y, \partial Y)$*.*

Proof. Consider a length-minimizing arc c of Y with endpoints in ∂Y which is not homotopic to an arc of ∂Y keeping its endpoints fixed. The arc c is a nonselfintersecting geodesic made of two minimizing segments of the same length meeting at a point x with $\mathrm{length}(c) = 2\,\mathrm{dist}(x, \partial Y)$. If c and Jc agree (up to orientation) then the arc c passes through a Weierstrass point (which agrees with x) and so $\mathrm{length}(c) \geq \beta$ from (4.1). We will therefore suppose otherwise.

By Proposition 3.1, the J-invariant simple loop $\delta \subset 3\mathbb{RP}^2$ of length at most $2C_{Bavard}$ lifts to a J-invariant simple loop in Y with the same length. This loop will still be denoted by δ.

By construction, the loops δ and ∂Y bound a cylinder in Y. In particular, the arc c intersects δ at exactly two points by minimality of c and δ, *cf.* [19]. These two points decompose c into three subarcs c', c_1 and c_2 with c' lying in $\Sigma_{1,1}$, *cf.* Figure 4.1, that is, $c = c_1 \cup c' \cup c_2$. Switching c_1 and c_2 if necessary, we can assume that c_1 is no longer than c_2.

The endpoints of c' and Jc', where J is the hyperelliptic involution on Y, decompose δ into four arcs a', Ja', b' and Jb', see Figure 4.1. Since c and δ are length-minimizing in their homotopy classes, the loop $a' \cup c'$ is noncontractible in $3\mathbb{RP}^2$ and so of length at least β. That is,

$$\mathrm{length}(a') + \mathrm{length}(c') \geq \beta. \tag{4.2}$$

By construction, the arc $c_1 \cup c' \cup b' \cup Jc_1$ with symmetric endpoints induces a noncontractible loop in $3\mathbb{RP}^2$. Indeed, its $\mathbb{Z}_2$-intersection

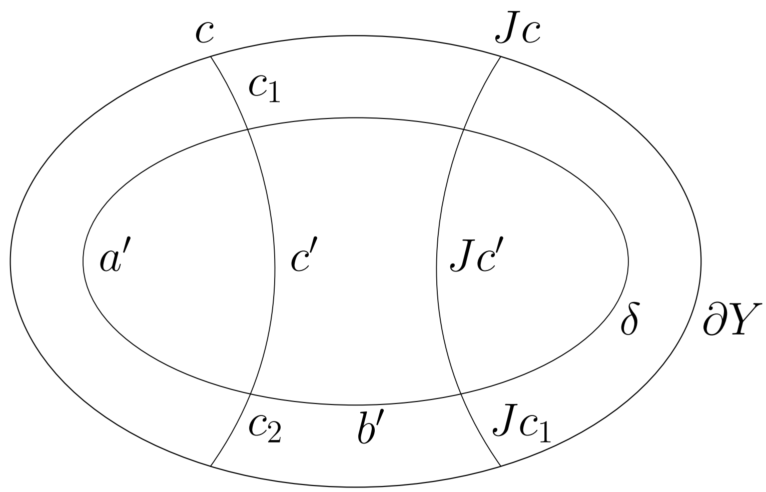

FIGURE 4.1. Intersection properties of c and δ

with the projection of ∂Y in $3\mathbb{RP}^2$ is nontrivial. Since the length of this arc is less or equal to the sum of the lengths of c and b', we obtain

$$\text{length}(c) + \text{length}(b') \geq \beta. \tag{4.3}$$

On the other hand, we have

$$\begin{aligned} \text{length}(a') + \text{length}(b') &= \frac{1}{2}\,\text{length}(\delta) \\ &\leq C_{Bavard} \end{aligned} \tag{4.4}$$

Thus, from (4.2), (4.3) and (4.4), we derive

$$\text{length}(c) \geq \frac{2\beta - C_{Bavard}}{2} \geq 0.6276.$$

□

We can now introduce the following definition.

Definition 4.5. Let U_h be the h-neighborhood of the equator in $3\mathbb{RP}^2$ with

$$h = \frac{1}{2}(\beta - \beta^{-1}) \simeq 0.144211.$$

From Lemma 4.4 (under the assumption that $\text{sys}(3\mathbb{RP}^2) \geq \beta$), the boundary components of U_h are formed of one loop freely homotopic to the equator and possibly several other contractible loops. Strictly speaking, we should first replace h with a nearby regular value of the distance function from the equator.

Define now $\hat{U}_h$ as the union of U_h and the disks bounded by its contractible boundary components. Note that $\hat{U}_h$ is a Möbius band which does not contain any Weierstrass point from Lemma 4.2

Proposition 4.6. *If* $\mathrm{sys}(3\mathbb{RP}^2) \geq \beta$ *then* $\mathrm{area}(\hat{U}_h) \geq 0.324$.

Proof. This area lower bound follows from an estimate due to Blatter [9, 10] and used by Sakai [45]. This is a lower bound for the area of a Möbius band in terms of its systole and length of path which is non-homotopic to the boundary, and with endpoints at the boundary circle. The lower bound equals half the area of a spherical belt formed by an h-neighborhood of the equator of a suitable sphere of constant curvature. Here the equator of the suitable sphere has length 2β and its radius is $r = \frac{2\beta}{2\pi} = \frac{\beta}{\pi}$. The antipodal quotient of the spherical belt is a Möbius band of systole β.

Let γ be the subtending angle γ of the northern half of the belt. Then γ satisfies

$$\gamma = \frac{h}{r} = \frac{h\pi}{\beta} = h\pi\beta^{-1} \geq 0.392403.$$

(here we use the values $h \geq 0.144211$ and $\beta^{-1} \geq 0.866133$). The height function is the moment map (Archimedes's theorem), and hence the area of the belt is proportional to $\sin\gamma \geq 0.382434$. The area of the corresponding region on the unit sphere is $4\pi \sin\gamma$. Hence the area of the spherical belt is $4\pi r^2 \sin\gamma = \frac{4\pi\beta^2 \sin\gamma}{\pi^2}$, which after quotienting by the antipodal map yields a lower bound

$$\mathrm{area}(\hat{U}_h) \geq \frac{2\beta^2 \sin\gamma}{\pi} = \frac{2(1.333)\sin\gamma}{\pi} \geq 0.324539,$$

proving the proposition. □

Let us show that the same lower bound holds for the area of the Möbius band Mob in $3\mathbb{RP}^2$. For this purpose, we will need the following result.

Lemma 4.7. *If* $\mathrm{sys}(3\mathbb{RP}^2) \geq \beta$ *then* $\mathrm{length}(\Delta) \leq \beta^{-1}$.

Proof. Let $d_{eq} : \hat{\mathbb{C}}^+ \to \mathbb{R}$ be the distance function from the equator of $\mathbb{C}^+$ endowed with the metric inherited from $3\mathbb{RP}^2$. The equator is at distance at least $\frac{1}{2}\beta$ from each of the three isolated branch points by Lemma 4.2. Each noncontractible connected component of the level curves of d_{eq} at distance at most $\frac{1}{2}\beta$ from the equator is longer than Δ. The coarea inequality gives a lower bound for the area of the strip S

formed by the noncontractible connected components of the level curves corresponding to distances

$$h \leq d_{eq} \leq \tfrac{1}{2}\beta.$$

By construction, this strip is disjoint from the projection of $\hat{U}_h$ on $\mathbb{C}^+$.

If length$(\Delta) \geq \beta^{-1}$, we obtain the following area lower bound for the strip:

$$\mathrm{area}(S) \geq \beta^{-1}\left(\tfrac{\beta}{2} - h\right) = \tfrac{1}{2}\beta^{-2} \geq 0.374.$$

Now, the strip S lifts to a region on $\Sigma_{1,1}$ of double the area, namely area at least 0.748, disjoint from $\hat{U}_h$. Combined with the lower bound of 0.324 for the area of $\hat{U}_h$ as in Proposition 4.6, this gives a total lower bound

$$\mathrm{area}(3\mathbb{RP}^2) \geq 2\ \mathrm{area}(S) + \mathrm{area}(\hat{U}_h) \geq 0.748 + 0.324 = 1.072$$

for the area of $3\mathbb{RP}^2$, which contradicts the original normalisation area$(3\mathbb{RP}^2) = 1$. □

Proposition 4.8. *If* sys$(3\mathbb{RP}^2) \geq \beta$ *then the loop* Δ *is at distance at least* h *from the equator. In particular,* area(Mob) ≥ 0.324.

Proof. By contraposition, we consider an arc γ of length less than h connecting Δ to the equator. By Lemma 4.7, we have length$(\Delta) \leq \beta^{-1}$. Then the path

$$\gamma \cup \Delta \cup \gamma^{-1}$$

produces a noncontractible loop on $3\mathbb{RP}^2$ of length less than

$$\mathrm{length}(\Delta) + 2h \leq \beta^{-1} + 2h = \beta,$$

proving the first statement of the proposition.

The second statement follows from Proposition 4.6 since the Möbius band Mob in $3\mathbb{RP}^2$ contains $\hat{U}_h$. □

We can now proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1. Suppose that sys$(3\mathbb{RP}^2) \geq \beta$. The surface $3\mathbb{RP}^2$ is separated into the union

$$3\mathbb{RP}^2 = \Sigma_{1,1} \cup \mathrm{Mob},$$

where area$(\Sigma_{1,1}) \leq 0.676$ and area(Mob) ≥ 0.324 from Proposition 4.8. Here, the separating loop is isometric to a circle twice as long as Δ, and therefore of radius

$$r = \frac{1}{\pi}\,\mathrm{length}(\Delta).$$

The area of a hemisphere based on such a circle is

$$2\pi r^2 = \frac{2}{\pi}\,\mathrm{length}(\Delta)^2 \leq \frac{2\beta^{-2}}{\pi}$$

by Lemma 4.7. Attaching the hemisphere to the torus with a disk removed produces a torus of total area at most $\frac{2}{\pi\beta^2} + 0.676$. Applying Loewner's bound (1.1) to the resulting torus, we obtain a systolic loop of square-length at most

$$\frac{2}{\sqrt{3}}\left(\frac{2}{\pi\beta^2} + 0.676\right) \leq 1.333,$$

proving the theorem. □

5. Other hyperelliptic surfaces

In this section, we prove Theorem 1.2 in the remaining case $n \geq 4$. Recall that a non-orientable surface is called *hyperelliptic* if its orientable double cover is.

Proposition 5.1. *Let $n \geq 4$. Every Riemannian metric from a hyperelliptic conformal type on the surface $n\mathbb{RP}^2$ satisfies the bound*

$$\frac{\mathrm{sys}^2}{\mathrm{area}} \leq \left(\frac{1}{4} + \frac{n}{8}\right)^{-1}.$$

In particular,

$$\mathrm{sys}^2 \leq 1.333\ \mathrm{area}.$$

Proof. Let $L = \mathrm{sys}(X)$. Without loss of generality, we can assume that the hyperelliptic invariant metric on X has the property that the area of every disk $B(R)$ of radius R with $R \leq L/2$ satisfies

$$\mathrm{area}(B(R)) \geq 2R^2, \tag{5.1}$$

see [33, Lemma 3.5].

For n even with $n = 2k \geq 4$, the orientable double cover of the surface $X = \Sigma_{k-1}\#K$ is a hyperelliptic surface Σ_{2k-1} of genus $2k-1$. As in Lemma 2.6, the preimage of the equatorial circle $\hat{\mathbb{R}} \subset \hat{\mathbb{C}}$ under the double cover $\Sigma_{2k-1} \to \hat{\mathbb{C}}$ is a pair of disjoint circles. This pair of circles bounds the preimage $Y \subset \Sigma_{2k-1}$ of the northern hemisphere $\hat{\mathbb{C}}^+$ under the previous double cover. The orientation-reversing involution τ on Σ_{2k-1} switches the two boundary components of Y. We can obtain X from Y by identifying the pairs of points of ∂Y corresponding to the orbits of the involution τ. Alternatively, we can define Y by compactifying the open surface $X \setminus \pi(\mathrm{Fix}(J \circ \tau))$, where $\pi : \Sigma_{2k-1} \to X$ is the quotient map induced by τ.

Let γ be a length-minimizing arc of Y joining the two boundary components of Y. The hyperelliptic involution J takes the endpoints of the 1-chain $\gamma \cup (-J\gamma)$ of Y to their opposite. As these endpoints lie in ∂Y, and since J and τ agree on ∂Y, the previous 1-chain of Y induces a loop c in X. This loop is noncontractible in X. Indeed, let X' be the complex with fundamental group isomorphic to $\mathbb{Z}$ obtained by collapsing the region of X outside a sufficiently small tubular neighborhood of the equator to a point. By construction, the loop c of X projects to a loop of X' representing twice a generator of $\pi_1 X' \simeq \mathbb{Z}$. Thus, c is noncontractible in X and so of length at least L. We deduce that the distance between the two boundary components of Y is at least $\frac{L}{2}$. Similarly, the distance between the Weierstrass points of Y and its boundary components is at least $\frac{L}{2}$, see Lemma 4.2. Likewise, the distance between any pair of Weierstrass points is at least $\frac{L}{2}$. This shows that the open disks D_i of radius $\frac{L}{4}$ centered at the $2k$ Weierstrass points of Y and the open $\frac{L}{4}$-tubular neighborhoods, U_1 and U_2, of the boundary components of Y are pairwise disjoint.

Now, as in Lemma 4.2, the level curves at distance at most $\frac{L}{4}$ from the boundary components of Y project to curves that separate the isolated branch points from the equator in $\hat{\mathbb{C}}^+$. Thus, each of these level curves has a noncontractible component in X and so is of length at least L. From the coarea inequality, we deduce that the area of each tubular neighborhood U_j is at least $\frac{L^2}{4}$. Since the metric satisfies (5.1), the area of D_i is at least $\frac{L^2}{8}$. Adding these lower bounds, we obtain

$$\operatorname{area}(X) \geq \left(\frac{1}{2} + \frac{n}{8}\right) L^2.$$

For n odd with $n = 2k+1 \geq 5$, the orientable double cover of $X = \Sigma_k \# \mathbb{RP}^2$ is a hyperelliptic surface Σ_{2k} of genus $2k$. As previously, the preimage of the equilatorial circle $\hat{\mathbb{R}} \subset \hat{\mathbb{C}}$ under the double cover $\Sigma_{2k} \to \hat{\mathbb{C}}$ is a single circle. This circle bounds the preimage Y of the northern hemisphere $\hat{\mathbb{C}}^+$ under the previous double cover. The orientation-reversing involution τ of Σ_{2k} takes the points of ∂Y to their opposite points. We can obtain X from Y by identifying the pairs of opposite points of ∂Y.

As previously, the distances between the Weierstrass points of Y and ∂Y, and between any pair of Weierstrass points are at least $\frac{L}{2}$.

Arguing as in the previous case, we deduce from the coarea inequality that the area of the $\frac{L}{4}$-tubular neighborhood of ∂Y is at least $\frac{L^2}{4}$. Combined with the estimates on the areas of the disjoint disks of radius $\frac{L}{4}$

centered at the Weierstrass points of Y, we obtain

$$\text{area}(X) \geq \left(\frac{1}{4} + \frac{n}{8}\right) L^2,$$

proving the proposition. □

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